Introduction to Logic (MA & Guests)
Q & A for Final exam; Practice exam 2019

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Question 1: Translation into propositional logic (10 points)

Translate the following sentences into *propositional logic*. Atomic sentences are represented by uppercase letters. Do not forget to provide the translation key — one key for the whole exercise. Represent as much logical structure as possible.

1. Libraries are nice study places if they are quiet places, and they are nice study places only if they are quiet places.
2. Either libraries are both nice study places and cosy meeting places, or they are quiet places.

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2. Either libraries are both nice study places and cosy meeting places, or they are quiet places.

Answer

Translation key:

- N: Libraries are nice study places.
- Q: Libraries are quiet places.
- C: Libraries are cosy meeting places.

Formalizations:

1. $(Q \to N) \land (N \to Q)$
   also okay: $Q \leftrightarrow N$ or $N \leftrightarrow Q$
2. $(N \land C) \lor Q$

Question 2: Translation into first-order logic (10 points)

Translate the following sentences to *first-order logic*. Do not forget to provide the translation key — one key for the whole exercise. Represent as much logical structure as possible.

1. All cities are linked to Rome and Rome is linked to all cities.
2. If there is a city that is more beautiful than Rome, then that city is either Venice or Florence, and no other city than that city is more beautiful than Rome.

Question 2: Translation into first-order logic (10 points)

Translate the following sentences to *first-order logic*. Do not forget to provide the translation key — one key for the whole exercise. Represent as much logical structure as possible.

1. All cities are linked to Rome and Rome is linked to all cities.
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Answer Translation key:

- $L(x, y)$: x is linked to y
- $B(x, y)$: x is is more beautiful than y
- a: Rome
- b: Florence
- c: Venice

Formalizations:

1. $\forall x L(x, a) \wedge \forall y L(a, y)$
2. $\forall x (B(x, a) \rightarrow ((x = b \vee x = c) \wedge \neg \exists y (y \neq x \wedge B(y, a))))$

Question 3: Formal proofs (30 points)

Give formal proofs of the following inferences. In items c and d, P, Q and R are predicate symbols. Do not forget the justifications. Only use the Introduction and Elimination rules and the Reiteration rule.

a
$$\begin{array}{|l} (A \land B) \lor (C \land D) \\ \hline (A \lor C) \land (B \lor D) \end{array}$$

b
$$\begin{array}{|l} A \leftrightarrow (\neg B \lor C) \\ \neg A \\ \hline B \land \neg C \end{array}$$

c
$$\begin{array}{|l} \forall x \forall y (Q(x) \rightarrow R(y)) \\ \hline \exists x Q(x) \rightarrow \forall y R(y) \end{array}$$

d
$$\begin{array}{|l} \exists y \exists x (x = y \land P(x, y)) \\ \hline \exists x P(x, x) \end{array}$$

Answer to Question 3 a

To prove:

$$(A \wedge B) \vee (C \wedge D) \vdash (A \vee C) \wedge (B \vee D)$$

First, set up the main structure:

1. $(A \wedge B) \vee (C \wedge D)$

$(A \vee C) \wedge (B \vee D)$

Answer to Question 3 a, continued

1.	$(A \wedge B) \vee (C \wedge D)$	
2	$A \wedge B$	
3	A	$\wedge$ Elim : 2
4	B	$\wedge$ Elim : 2
5	$A \vee C$	$\vee$ Intro : 3
6	$B \vee D$	$\vee$ Intro : 4
7.	$(A \vee C) \wedge (B \vee D)$	$\wedge$ Intro : 5, 6
8	$C \wedge D$	
9	C	$\wedge$ Elim : 8
10	D	$\wedge$ Elim : 8
11	$A \vee C$	$\vee$ Intro : 9
12	$B \vee D$	$\vee$ Intro : 10
13	$(A \vee C) \wedge (B \vee D)$	$\wedge$ Intro : 11, 12
14	$(A \vee C) \wedge (B \vee D)$	$\vee$ Elim : 1, 2-7, 8-13

Answer to Question 3 b

To prove:

$$\begin{array}{|l} A \leftrightarrow (\neg B \vee C) \\ \neg A \\ \hline B \wedge \neg C \end{array}$$

First, set up the main structure:

$$\begin{array}{|l} 1.\ A \leftrightarrow (\neg B \vee C) \\ 2.\ \neg A \\ \hline \\ \\ \\ B \wedge \neg C \end{array}$$

Answer to Question 3b, continued

1	$A \leftrightarrow (\neg B \vee C)$	
2	$\neg A$	
3	$\neg B$	
4	$\neg B \vee C$	$\vee$ Intro : 3
5	A	$\leftrightarrow$ Elim: 1, 4
6	$\bot$	$\bot$ Intro : 5, 2
7	$\neg\neg B$	$\neg$ Intro : 3-6
8	B	$\neg$ Elim : 7
9	C	
10	$\neg B \vee C$	$\vee$ Intro : 9
11	A	$\leftrightarrow$ Elim : 1, 10
12	$\bot$	$\bot$ Intro : 11, 2
13	$\neg C$	$\neg$ Intro : 9-12
14	$B \wedge \neg C$	$\wedge$ Intro : 8, 13

Answer to Question 3 c

To prove: $\forall x \forall y(Q(x) \rightarrow R(y))$ ⊢ $\exists x Q(x) \rightarrow \forall y R(y)$

First, set up the main structure:

1. $\forall x \forall y(Q(x) \rightarrow R(y))$

$\exists x Q(x) \rightarrow \forall y R(y)$

Answer to Question 3 c, continued

1	$\forall x \forall y (Q(x) \to R(y))$	
2	$\exists x Q(x)$	
3	$\boxed{a}$ $Q(a)$	
4	$\boxed{c}$	
5	$\forall y (Q(a) \to R(y))$	$\forall$ Elim: 1
6	$Q(a) \to R(c)$	$\forall$ Elim: 5
7	$R(c)$	$\to$ Elim: 6, 3
8	$\forall y R(y)$	$\forall$ Intro: 4–7
9	$\forall y R(y)$	$\exists$ Elim: 2, 3–8
10	$\exists x Q(x) \to \forall y R(y)$	$\to$ Intro: 2–9

Answer to Question 3 d

To prove: $\exists y \exists x(x = y \wedge P(x, y)) \vdash \exists x P(x, x)$

First, set up the main structure:

1. $\exists y \exists x(x = y \wedge P(x, y))$

$\exists x P(x, x)$

Answer to Question 3 d, continued

1	$\exists x \exists y (x = y \wedge P(x,y))$	
2	$\boxed{a}$ $\exists y (a = y \wedge P(a,y))$	
3	$\boxed{c}$ $(a = c) \wedge P(a,c)$	
4.	$a = c$	$\wedge$ Elim: 3
5.	$P(a,c)$	$\wedge$ Elim: 3
6.	$P(c,c)$	$=$ Elim: 5,4
7.	$\exists x\, P(x,x)$	$\exists$ Intro: 6
8.	$\exists x\, P(x,x)$	$\exists$ Elim: 2, 3-7
9.	$\exists x\, P(x,x)$	$\exists$ Elim: 1, 2-8

Question 4: Truth tables (15 points)

Use *truth tables* to answer the next questions. Make the full truth tables, and do not forget to draw conclusions from the truth tables in order to explain your answers. Order the rows as follows:

P	Q	R	…
T	T	T	…
T	T	F	…
T	F	T	…
T	F	F	…
F	T	T	…
F	T	F	…
F	F	T	…
F	F	F	…

Small(a)	Medium(b)	Smaller(a, b)	…
T	T	T	…
T	T	F	…
T	F	T	…
T	F	F	…
F	T	T	…
F	T	F	…
F	F	T	…
F	F	F	…

a: Check with a truth table whether the following formula is a tautology. $(A \vee (B \to C)) \to (B \vee (A \to C))$

b: Check with a truth table whether the following formula is Tarski's World-(TW-)possible. Indicate clearly which rows are spurious (if any). $(\mathrm{Small(a)} \leftrightarrow \mathrm{Medium(b)}) \wedge \neg(\mathrm{Smaller(a,b)} \vee \neg\mathrm{Medium(b)})$

Answer to Question 4a

A	B	C	$(A$	$\vee$	$(B \rightarrow C)$	$)$	$\rightarrow$	$(B$	$\vee$	$(A \rightarrow C)$	$)$
T	T	T		T	T		T		T	T	
T	T	F		T	F		T		T	F	
T	F	T		T	T		T		T	T	
T	F	F		T	T		F		F	F	
F	T	T		T	T		T		T	T	
F	T	F		F	F		T		T	T	
F	F	T		T	T		T		T	T	
F	F	F		T	T		T		T	T	
				2.	1.		3.		2.	1.	

There is a row with an 'F' under the main connective, namely row 4. Hence this formula is *not a tautology*.

Answer to Question 4b

Small(a)	Medium(b)	Smaller(a,b)	(Small(a) $\leftrightarrow$ Medium(b))	$\wedge$	$\neg$ (Smaller(a,b)	$\vee$	$\neg$ Medium(b)
T	T	T	T	F	F	T	F
T	T	F	T	T	T	F	F
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	F	F	F	T	F
F	T	F	F	F	T	F	F
F	F	T	T	F	F	T	T
F	F	F	T	F	F	T	T
			1.	4.	3.	2.	1.

Rows 2 and 5 are spurious. There is only one ‘T’ under the main connective in the second row, but this row is spurious. Hence this formula is *not TW-possible*.

Question 5: Normal forms propositional logic (5 points)

Conjunctive normal form

Provide a conjunctive normal form (CNF) of the following formula. Show all of the intermediate steps.

$$(\neg C \land B) \leftrightarrow \neg(A \lor \neg B)$$

Answer to Question 5

Provide a conjunctive normal form (CNF) of the following formula. Show all of the intermediate steps.

$(\neg C \land B) \leftrightarrow \neg(A \lor \neg B)$

$\Leftrightarrow ((\neg C \land B) \rightarrow \neg(A \lor \neg B)) \land (\neg(A \lor \neg B) \rightarrow (\neg C \land B))$

by unraveling $\leftrightarrow$

$\Leftrightarrow (\neg(\neg C \land B) \lor \neg(A \lor \neg B)) \land ((\neg\neg A \lor \neg B) \lor (\neg C \land B))$

by unraveling $\rightarrow$

$\Leftrightarrow (\neg(\neg C \land B) \lor \neg(A \lor \neg B)) \land ((A \lor \neg B) \lor (\neg C \land B))$

by double negation deletion

$\Leftrightarrow ((\neg\neg C \lor \neg B) \lor (\neg A \land \neg\neg B)) \land ((A \lor \neg B) \lor (\neg C \land B))$

by deMorgan

$\Leftrightarrow ((C \lor \neg B) \lor (\neg A \land B)) \land ((A \lor \neg B) \lor (\neg C \land B))$

by double negation deletion

$\Leftrightarrow (C \lor \neg B \lor \neg A) \land (C \lor \neg B \lor B) \land (A \lor \neg B \lor \neg C) \land (A \lor \neg B \lor B)$

by distributivity

The last formula above is already a correct CNF but it is extra nice to simplify even further:

$\Leftrightarrow (C \lor \neg B \lor \neg A) \land (A \lor \neg B \lor \neg C)$ by $\neg B \lor B \Leftrightarrow \top$

Question 6: Normal forms for first-order logic and Horn sentences (10 points)

Question 6

a) Provide a Prenex normal form of the following sentence. Show all intermediate steps.

$$\exists x \forall y R(y,x) \vee \neg \forall z \exists x \exists y (Q(z,y,x) \wedge \neg P(x))$$

b) Provide a Skolem normal form of the following sentence. Show all intermediate steps.

$$\forall x \forall y \exists z \forall w \exists v ((R(x,z) \rightarrow Q(w,v,z)) \vee (\neg R(y,z) \wedge P(w)))$$

c) Check the satisfiability of the following Horn sentence.

$$A \wedge \neg B \wedge (\neg A \vee B \vee \neg C) \wedge (C \vee \neg E) \wedge (\neg A \vee \neg D \vee E)$$

Use the Horn algorithm and indicate the order in which you assign truth values to the atomic sentences. If you prefer the conditional form, you may also rewrite the formula with $\rightarrow$ and then use the satisfiability algorithm for conditional Horn sentences.

Answer to Question 6a

$$
\begin{array}{ll}
 & \exists x \forall y R(y,x) \vee \neg \forall z \exists x \exists y (Q(z,y,x) \wedge \neg P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \neg \exists x \exists y (Q(z,y,x) \wedge \neg P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \forall x \neg \exists y (Q(z,y,x) \wedge \neg P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \forall x \forall y \neg (Q(z,y,x) \wedge \neg P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \forall x \forall y (\neg Q(z,y,x) \vee \neg\neg P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \forall x \forall y (\neg Q(z,y,x) \vee P(x)) \\
\Leftrightarrow & \exists x \forall y R(y,x) \vee \exists z \forall v \forall w (\neg Q(z,w,v) \vee P(v)) \\
\Leftrightarrow & \exists x \forall y \exists z \forall v \forall w (R(y,x) \vee \neg Q(z,w,v) \vee P(v))
\end{array}
$$

The resulting formula is in prenex normal form.

Answer to Question 6b

$\forall x \forall y \exists z \forall w \exists v ((R(x,z) \to Q(w,v,z)) \vee (\neg R(y,z) \wedge P(w)))$

(replace z with a binary function $f(x,y)$)

$\forall x \forall y \forall w \exists v ((R(x,f(x,y)) \to$
$Q(w,v,f(x,y))) \vee (\neg R(y,f(x,y)) \wedge P(w)))$

(replace v with a ternary function $g(x,y,w)$)

$\forall x \forall y \forall w ((R(x,f(x,y)) \to$
$Q(w,g(x,y,w),f(x,y))) \vee (\neg R(y,f(x,y)) \wedge P(w)))$

The resulting formula is in Skolem normal form.

Answer to Question 6 c

Result:

A	B	C	D	E	$A\wedge$	$\neg B\wedge($	$\neg A\vee$	B	$\vee\neg C)\wedge($	$C\vee$	$\neg E)\wedge($	$\neg A\vee$	$\neg D\vee$	$E)$
T	F	F	F	F	T	T	F	F	T	F	T	F	T	F

Order of assignments:
first $A := \mathrm{T}$;
then all at once: $B := \mathrm{F}$, $C := \mathrm{F}$, $D := \mathrm{F}$, and $E := \mathrm{F}$ (step 4 of the algorithm).
All conjuncts can be made true. The formula is therefore satisfiable.

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1, 2\}$, $C = \{\langle 1, 2\rangle\}$, $D = \{2, \langle 2, 2\rangle\}$, and $R = \{\langle 2, 2\rangle, \langle 1, 2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$

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2. $A \subseteq C$

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1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$

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2. $A \subseteq C$ No
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For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1,2\}$, $C = \{\langle 1,2\rangle\}$, $D = \{2, \langle 2,2\rangle\}$, and $R = \{\langle 2,2\rangle, \langle 1,2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

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For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1,2\}$, $C = \{\langle 1,2\rangle\}$, $D = \{2, \langle 2,2\rangle\}$, and $R = \{\langle 2,2\rangle, \langle 1,2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

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For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

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2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$

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For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$ Yes

Question 7: Set theory (10 points)

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For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$ Yes
7. $\emptyset = D \cap C$

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1,2\}$, $C = \{\langle 1,2\rangle\}$, $D = \{2, \langle 2,2\rangle\}$, and $R = \{\langle 2,2\rangle, \langle 1,2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

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5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$ Yes
7. $\emptyset = D \cap C$ Yes

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1, 2\}$, $C = \{\langle 1, 2\rangle\}$, $D = \{2, \langle 2, 2\rangle\}$, and $R = \{\langle 2, 2\rangle, \langle 1, 2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$ Yes
7. $\emptyset = D \cap C$ Yes
8. $B \setminus A \in C$

Question 7: Set theory (10 points)

Given are the following five sets: $A = \{2\}$, $B = \{1,2\}$, $C = \{\langle 1,2\rangle\}$, $D = \{2, \langle 2,2\rangle\}$, and $R = \{\langle 2,2\rangle, \langle 1,2\rangle\}$. For each of the following statements, determine whether it is true or false. You are not required to explain the answer.

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1. $\emptyset \in A$ No
2. $A \subseteq C$ No
3. $A \subset D$ Yes
4. $(B \setminus C) \cup A = B$ Yes
5. $A \subset (R \cap C) \cup B$ Yes
6. $(R \setminus C) \cup (A \setminus C) = D$ Yes
7. $\emptyset = D \cap C$ Yes
8. $B \setminus A \in C$ No

Question 8: Translating function symbols (7 points)

Translate the following sentences using the translation key provided.

The domain of discourse is the set of all persons.

a: Russell
b: Wittgenstein
supervisor(x): x's supervisor
StudentOf(x, y): x is a student of y

a: Russell is the supervisor of Wittgenstein, but Wittgenstein is nobody's supervisor.
b: For each pair of two persons, the first person is the supervisor of the second one if and only if the second person is a student of the first person.
c: Every supervisor of a supervisor is a student of someone.

Answers Question 8

a: Russell is the supervisor of Wittgenstein, but Wittgenstein is nobody's supervisor.

Answers Question 8

a: Russell is the supervisor of Wittgenstein, but Wittgenstein is nobody's supervisor.
$a = \text{supervisor}(b) \wedge \neg\exists x(b = \text{supervisor}(x))$

Answers Question 8

a: Russell is the supervisor of Wittgenstein, but Wittgenstein is nobody's supervisor.
$a = \text{supervisor}(b) \wedge \neg\exists x(b = \text{supervisor}(x))$

b: For each pair of two persons, the first person is the supervisor of the second one if and only if the second person is a student of the first person.

Answers Question 8

a: Russell is the supervisor of Wittgenstein, but Wittgenstein is nobody's supervisor.
$a = \text{supervisor}(b) \wedge \neg\exists x(b = \text{supervisor}(x))$

b: For each pair of two persons, the first person is the supervisor of the second one if and only if the second person is a student of the first person.
$\forall x \forall y(x = \text{supervisor}(y) \leftrightarrow \text{StudentOf}(y, x))$
Also correct: $\forall x \forall y(x \neq y \rightarrow (x = \text{supervisor}(y) \leftrightarrow \text{StudentOf}(y, x)))$

c: Every supervisor of a supervisor is a student of someone.

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Also correct: $\forall x \forall y(x \neq y \rightarrow (x = \text{supervisor}(y) \leftrightarrow \text{StudentOf}(y, x)))$

c: Every supervisor of a supervisor is a student of someone.
$\forall x \exists z(\text{StudentOf}(\text{supervisor}(\text{supervisor}(x)), z))$
alternative:
$\forall x(\exists y(x = \text{supervisor}(\text{supervisor}(y))) \rightarrow \exists z \text{StudentOf}(x, z))$
alternative:
$\forall x \forall y(x = \text{supervisor}(\text{supervisor}(y)) \rightarrow \exists z \text{StudentOf}(x, z))$

Question 9: Semantics (10 points)

Let a model $\mathfrak{M}$ with domain $\mathfrak{M}(\forall) = \{1, 2\}$ be given such that

- $\mathfrak{M}(a) = 1$
- $\mathfrak{M}(P) = \{1\}$
- $\mathfrak{M}(R) = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$

Let h be an assignment such that:

- $h(x) = 1$
- $h(y) = 2$

Evaluate the following statements. Follow the truth definition step by step. a) $\mathfrak{M} \models R(a, x) \vee (R(y, y) \wedge \neg P(a))[h]$
b) $\mathfrak{M} \models \exists x \forall y (R(x, y) \rightarrow P(y))[h]$
c) $\mathfrak{M} \models \forall x (P(x) \rightarrow \exists y (\neg P(y) \wedge R(y, y)))[h]$

Answer to Question 9a

$\mathfrak{M} \models R(a,x) \vee (R(y,y) \wedge \neg P(a))[h]$
By the semantics of $\vee$ the statement is equivalent to:
(1) $\mathfrak{M} \models R(a,x)[h]$ or (2) $\mathfrak{M} \models R(y,y) \wedge \neg P(a)[h]$.
We consider each disjunct in turn.
Disjunct (1) is equivalent to $\left\langle [\![a]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \right\rangle \in \mathfrak{M}(R)$, which in turn is equivalent to $\langle \mathfrak{M}(a), h(x) \rangle \in \mathfrak{M}(R)$, that is,
$\langle 1,1 \rangle \in \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$, which is false.

Answer to Question 9a

$\mathfrak{M} \models R(a,x) \vee (R(y,y) \wedge \neg P(a))[h]$
By the semantics of $\vee$ the statement is equivalent to:
(1) $\mathfrak{M} \models R(a,x)[h]$ or (2) $\mathfrak{M} \models R(y,y) \wedge \neg P(a)[h]$.
We consider each disjunct in turn.
Disjunct (1) is equivalent to $\langle [\![a]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$, which in turn is equivalent to $\langle \mathfrak{M}(a), h(x) \rangle \in \mathfrak{M}(R)$, that is,
$\langle 1,1 \rangle \in \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$, which is false.
Disjunct (2), by the semantics of $\wedge$ and $\neg$ is equivalent to
(2.1) $\mathfrak{M} \models R(y,y)[h]$ and (2.2) $\mathfrak{M} \not\models P(a)[h]$.
Conjunct (2.1) is equivalent to the statement $\langle [\![y]\!]_h^{\mathfrak{M}}, [\![y]\!]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$, which in turn is equivalent to $\langle h(y), h(y) \rangle \in \mathfrak{M}(R)$, that is,
$\langle 2,2 \rangle \in \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$, which is true.
Conjunct (2.2) is equivalent to the statement $[\![a]\!]_h^{\mathfrak{M}} \notin \mathfrak{M}(P)$, which in turn is equivalent to $\mathfrak{M}(a) \notin \mathfrak{M}(P)$, that is $1 \notin \{1\}$, which is false.
Disjunct (2) is therefore false.

Answer to Question 9a

$\mathfrak{M} \models R(a,x) \vee (R(y,y) \wedge \neg P(a))[h]$

By the semantics of $\vee$ the statement is equivalent to:

(1) $\mathfrak{M} \models R(a,x)[h]$ or (2) $\mathfrak{M} \models R(y,y) \wedge \neg P(a)[h]$.

We consider each disjunct in turn.

Disjunct (1) is equivalent to $\langle [\![a]\!]_h^{\mathfrak{M}}, [\![x]\!]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$, which in turn is equivalent to $\langle \mathfrak{M}(a), h(x) \rangle \in \mathfrak{M}(R)$, that is, $\langle 1,1 \rangle \in \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$, which is false.

Disjunct (2), by the semantics of $\wedge$ and $\neg$ is equivalent to (2.1) $\mathfrak{M} \models R(y,y)[h]$ and (2.2) $\mathfrak{M} \not\models P(a)[h]$.

Conjunct (2.1) is equivalent to the statement $\langle [\![y]\!]_h^{\mathfrak{M}}, [\![y]\!]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(R)$, which in turn is equivalent to $\langle h(y), h(y) \rangle \in \mathfrak{M}(R)$, that is, $\langle 2,2 \rangle \in \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$, which is true.

Conjunct (2.2) is equivalent to the statement $[\![a]\!]_h^{\mathfrak{M}} \notin \mathfrak{M}(P)$, which in turn is equivalent to $\mathfrak{M}(a) \notin \mathfrak{M}(P)$, that is $1 \notin \{1\}$, which is false.

Disjunct (2) is therefore false.

Both disjuncts (1) and (2) are false, so the original statement is false.

Answer to Question 9b

$\mathfrak{M} \models \exists x \forall y(R(x,y) \to P(y))[h]$
By the semantics of $\exists$ the statement is equivalent to:
there exists $d \in \mathfrak{M}(\forall)$ s.t. $\mathfrak{M} \models \forall y(R(x,y) \to P(y))[h[x/d]]$.
By the semantics of $\forall$, this is in turn equivalent to:
there exists $d \in \mathfrak{M}(\forall)$ s.t. for all $e \in \mathfrak{M}(\forall)$,
$\mathfrak{M} \models R(x,y) \to P(y)[h[x/d][y/e]]$.
By the semantics of $\to$, this is in turn equivalent to:
there exists $d \in \mathfrak{M}(\forall)$ s.t. for all $e \in \mathfrak{M}(\forall)$,
$\mathfrak{M} \not\models R(x,y)[h[x/d][y/e]]$ or $\mathfrak{M} \models P(y)[h[x/d][y/e]]$. This is equivalent to:
(1) there exists $d \in \mathfrak{M}(\forall)$ s.t. for all $e \in \mathfrak{M}(\forall)$,
$\left\langle [\![x]\!]^{\mathfrak{M}}_{h[x/d][y/e]}, [\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \right\rangle \notin \mathfrak{M}(R)$ or $[\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \in \mathfrak{M}(P)$.
There are two cases, either of which should be true for the original statement to be true: $d = 1$ or $d = 2$. We consider them in turn.

Answer to Question 9b, continued

$[d = 1]$ Under this assumption (1) is equivalent to: (2) for all $e \in \mathfrak{M}(\forall)$, $\left\langle 1, [\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \right\rangle \notin \mathfrak{M}(R)$ or $[\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \in \mathfrak{M}(P)$. There are two cases, both of which should be true for the latter statement to be true: $e = 1$ and $e = 2$. If $e = 1$, (2) is equivalent to (2.1) $\langle 1,1 \rangle \notin \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$ or (2.2) $1 \in \{1\}$. Both are true, hence if $e = 1$ (2) is true. If $e = 2$, (2) is equivalent to (2.1') $\langle 1,2 \rangle \notin \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,2 \rangle\}$ or (2.2') $2 \in \{1\}$. Both are false, hence if $e = 2$ (2) is false. We conclude that if $d = 1$, (1) is false.

Answer to Question 9b, continued

$[d=1]$ Under this assumption (1) is equivalent to: (2) for all $e \in \mathfrak{M}(\forall)$, $\left\langle 1, [\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \right\rangle \notin \mathfrak{M}(R)$ or $[\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \in \mathfrak{M}(P)$. There are two cases, both of which should be true for the latter statement to be true: $e=1$ and $e=2$. If $e=1$, (2) is equivalent to (2.1) $\langle 1,1\rangle \notin \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ or (2.2) $1 \in \{1\}$. Both are true, hence if $e=1$ (2) is true. If $e=2$, (2) is equivalent to (2.1') $\langle 1,2\rangle \notin \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ or (2.2') $2 \in \{1\}$. Both are false, hence if $e=2$ (2) is false. We conclude that if $d=1$, (1) is false.

$[d=2]$ Under this assumption (1) is equivalent to: (2') for all $e \in \mathfrak{M}(\forall)$, $\left\langle 2, [\![y]\!]^{\mathfrak{M}}_{h[x/2][y/e]} \right\rangle \notin \mathfrak{M}(R)$ or $[\![y]\!]^{\mathfrak{M}}_{h[x/2][y/e]} \in \mathfrak{M}(P)$. There are two cases, both of which should be true for the latter statement to be true: $e=1$ and $e=2$. If $e=1$, (2') is equivalent to (2'.1) $\langle 2,1\rangle \notin \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ or (2'.2) $2 \in \{1\}$. (2'.1) is true, hence if $e=1$ (2') is true. If $e=2$, (2) is equivalent to (2'.1') $\langle 2,2\rangle \notin \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ or (2'.2') $2 \in \{1\}$. Both are false, hence if $e=2$ (2') is false. We conclude that if $d=2$, (1) is false.

It follows that the original statement is false.

Answer to Question 9c

$\mathfrak{M} \models \forall x(P(x) \to \exists y(\neg P(y) \wedge R(y,y)))[h]$
By the semantics of $\forall$ the statement is equivalent to:
for all $d \in \mathfrak{M}(\forall)$, $\mathfrak{M} \models P(x) \to \exists y(\neg P(y) \wedge R(y,y))[h[x/d]]$.
By the semantics of $\to$ this is in turn equivalent to:
for all $d \in \mathfrak{M}(\forall)$, (1) $\mathfrak{M} \not\models P(x)[h[x/d]]$ or
(2) $\mathfrak{M} \models \exists y(\neg P(y) \wedge R(y,y))[h[x/d]]$.
(1) is equivalent to $[\![x]\!]^{\mathfrak{M}}_{h[x/d]} \notin \mathfrak{M}(P)$.
By the semantics of $\exists$ (2) is equivalent to:
there exists $e \in \mathfrak{M}(\forall)$ s.t. $\mathfrak{M} \models \neg P(y) \wedge R(y,y)[h[x/d][y/e]]$.
By the semantics of $\wedge$ and $\neg$ the latter statement is equivalent to:
there exists $e \in \mathfrak{M}(\forall)$ s.t. (2.1) $\mathfrak{M} \not\models P(y)[h[x/d][y/e]]$ and (2.2) $\mathfrak{M} \models R(y,y)[h[x/d][y/e]]$. (2.1) is equivalent to
$[\![x]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \notin \mathfrak{M}(P)$ and (2.2) to
$\left\langle [\![y]\!]^{\mathfrak{M}}_{h[x/d][y/e]}, [\![x]\!]^{\mathfrak{M}}_{h[x/d][y/e]} \right\rangle \in \mathfrak{M}(R)$.
There are two cases, both of which should be true for the original statement to be true: $d = 1$ and $d = 2$.

Answer to Question 9c, Continued

$[d = 1]$ Under this assumption, the original statement is equivalent to: $1 \notin \{1\}$ or there exists $e \in \mathfrak{M}(\forall)$ s.t. $[\![1]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \notin \mathfrak{M}(P)$ and $\left\langle [\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]}, [\![x]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \right\rangle \in \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. The first disjunct is false. The second disjunct is true for $e = 2$. Under that assumption, it is equivalent to $\langle 2,2\rangle \in \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. So if $d = 1$, the original statement is true and we do not need to consider the case of $e = 1$.

Answer to Question 9c, Continued

$[d = 1]$ Under this assumption, the original statement is equivalent to: $1 \notin \{1\}$ or there exists $e \in \mathfrak{M}(\forall)$ s.t. $[\![1]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \notin \mathfrak{M}(P)$ and $\left\langle [\![y]\!]^{\mathfrak{M}}_{h[x/1][y/e]}, [\![x]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \right\rangle \in \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. The first disjunct is false. The second disjunct is true for $e = 2$. Under that assumption, it is equivalent to $\langle 2,2\rangle \in \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. So if $d = 1$, the original statement is true and we do not need to consider the case of $e = 1$.

$[d = 2]$ Under this assumption, the original statement is equivalent to: $2 \notin \{1\}$ or there exists $e \in \mathfrak{M}(\forall)$ s.t. $[\![1]\!]^{\mathfrak{M}}_{h[x/2][y/e]} \notin \mathfrak{M}(P)$ and $\left\langle [\![y]\!]^{\mathfrak{M}}_{h[x/2][y/e]}, [\![x]\!]^{\mathfrak{M}}_{h[x/1][y/e]} \right\rangle \in \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$. The first disjunct is true. So if $d = 2$ the original statement is true and we do not need to consider the second disjunct.
It follows that the original statement is true.

Question 10: Bonus question (10 points)

Give a formal proof of the following inference. Don't forget to provide justifications. Only use the Introduction and Elimination rules and the Reiteration rule.

$$\begin{array}{|l} \neg\forall x \exists y \forall z \neg R(x,y,z) \\ \hline \exists u \forall v \exists w R(u,v,w) \end{array}$$

Answer bonus Question

1	$\neg\forall x\, \exists y\, \forall z\, \neg R(x,y,z)$	
2	$\neg\exists u\, \forall v\, \exists w\, R(u,v,w)$	
3	$\boxed{c}$	
4	$\neg\exists y\, \forall z\, \neg R(c,y,z)$	
5	$\boxed{d}$	
6	$\neg\exists w\, R(c,d,w)$	
7	$\boxed{e}$	
8	$R(c,d,e)$	
9	$\exists w\, R(c,d,w)$	$\exists$- Intro: 8
10	$\perp$	$\perp$ Intro: 9, 6
11	$\neg R(c,d,e)$	$\neg$ Intro: 8–10
12	$\forall z\, \neg R(c,d,z)$	$\forall$ Intro: 7–11
13	$\exists y\, \forall z\, \neg R(c,y,z)$	$\exists$ Intro: 12
14	$\perp$	$\perp$ Intro: 13, 4
15	$\neg\neg\exists w\, R(c,d,w)$	$\neg$ Intro: 6–14
16	$\exists w\, R(c,d,w)$	$\neg$ Elim: 15
17	$\forall v\, \exists w\, R(c,v,w)$	$\forall$ Intro: 5–16
18	$\exists u\, \forall v\, \exists w\, R(u,v,w)$	$\exists$ Intro: 17
19	$\perp$	$\perp$ Intro: 18, 2
20	$\neg\neg\exists y\, \forall z\, \neg R(c,y,z)$	$\neg$ Intro: 4–19
21	$\exists y\, \forall z\, \neg R(c,y,z)$	$\neg$ Elim: 20
22	$\forall x\, \exists y\, \forall z\, \neg R(x,y,z)$	$\forall$ Intro: 3–21
23	$\perp$	$\perp$ Intro: 22, 1
24	$\neg\neg\exists u\, \forall v\, \exists w\, R(u,v,w)$	$\neg$ Intro: 2–23
25	$\exists u\, \forall v\, \exists w\, R(u,v,w)$	$\neg$ Elim: 24